Section 3.5 Limits at Infinity

Limits at Infinity

This section discusses the "end behavior" of a function on an *infinite* interval. Consider the graph of

$$f(x) = \frac{3x^2}{x^2 + 1}$$

as shown in Figure 3.33. Graphically, you can see that the values of $f(x)$ appear to approach 3 as x increases without bound or decreases without bound. You can come to the same conclusions numerically, as shown in the table.

← x decreases without bound. | x increases without bound. →

x	$-\infty \leftarrow$	-100	-10	-1	0	1	10	100	$\rightarrow \infty$
$f(x)$	$3 \leftarrow$	2.9997	2.97	1.5	0	1.5	2.97	2.9997	$\rightarrow 3$

← $f(x)$ approaches 3. | $f(x)$ approaches 3. →

BAD FORM

WRONG

Illegal

The table suggests that the value of $f(x)$ approaches 3 as x increases without bound $(x \to \infty)$. Similarly, $f(x)$ approaches 3 as x decreases without bound $(x \to -\infty)$. These **limits at infinity** are denoted by

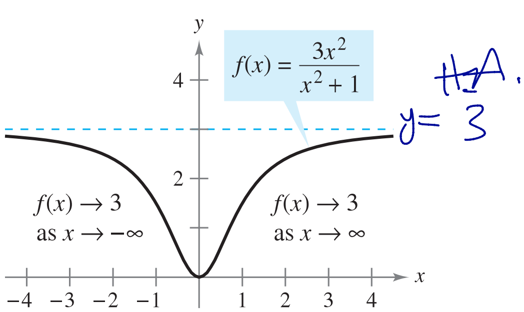
$$\lim_{x\to\infty} \frac{3x^2}{x^2+1} = \frac{3\cdot(\infty)^2}{(\infty)^2+1} = \frac{3\cdot\infty}{\infty+1} = \frac{3\cdot\infty}{\infty}$$

$$\boxed{y = 3}$$

$$\lim_{x\to-\infty} f(x) = 3$$ Limit at negative infinity

and

$$\lim_{x\to\infty} f(x) = 3.$$ Limit at positive infinity



The limit of $f(x)$ as x approaches $-\infty$ or ∞ is 3.

Figure 3.33

To say that a statement is true as x increases *without bound* means that for some (large) real number M, the statement is true for *all* x in the interval $\{x: x > M\}$. The following definition uses this concept.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

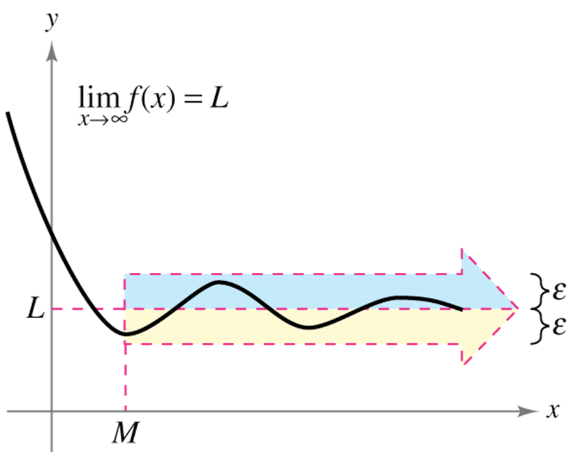


Figure 3.34

NOTE The statement $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$ means that the limit exists *and* the limit is equal to L.

Horizontal Asymptotes

In Figure 3.34, the graph of f approaches the line $y = L$ as x increases without bound. The line $y = L$ is called a **horizontal asymptote** of the graph of f.

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x\to-\infty} f(x) = L \quad \text{or} \quad \lim_{x\to\infty} f(x) = L.$$

Note that from this definition, it follows that the graph of a *function* of x can have at most two horizontal asymptotes—one to the right and one to the left.

Limits at infinity have many of the same properties of limits discussed in Section 1.3. For example, if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ both exist, then

$$\lim_{x\to\infty} [f(x) + g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

and

$$\lim_{x\to\infty} [f(x)g(x)] = \left[\lim_{x\to\infty} f(x)\right]\left[\lim_{x\to\infty} g(x)\right].$$

THEOREM 3.10 Limits at Infinity

If r is a positive rational number and c is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

Ex.1 Find the limit: $\lim\limits_{x\to\infty}\left(4+\dfrac{3}{x^2+1}\right)$

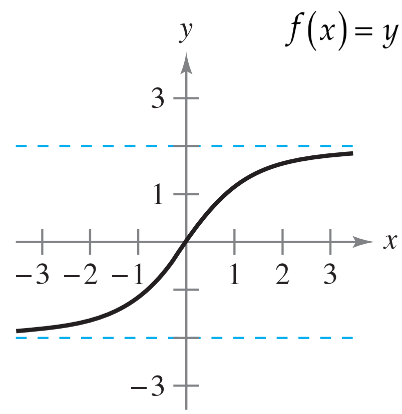
$$\lim_{x\to\infty}\left(4+\frac{3}{x^2+1}\right)$$

$$= \lim_{x\to\infty} 4 + \lim_{x\to\infty}\frac{3}{x^2+1}$$

$$= 4 + 0$$

$$= 4$$

Ex.2 Given the follow graph of f, find the two limits:



(a) $\lim\limits_{x\to\infty} f(x) =$

(b) $\lim\limits_{x\to-\infty} f(x) =$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exist.

Ex.3 Find the limit: $\lim\limits_{x\to\infty} \dfrac{x^2+3}{2x^2-1}$

$$\lim_{x\to\infty} \frac{x^2+3}{2x^2-1}$$

$$= \lim_{x\to\infty} \left[\frac{x^2+3}{2x^2-1}\right]\left[\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{3}{x^2}}{\frac{2x^2}{x^2}-\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\left(1+\frac{3}{x^2}\right)}{\lim_{x\to\infty}\left(2-\frac{1}{x^2}\right)}$$

$$= \frac{1+0}{2-0}$$

$$= \frac{1}{2}$$

Thinking Informal

$$\lim_{x\to\infty}\frac{5x^{3/2}}{4x^2+1} \approx \lim_{x\to\infty}\frac{5x^{3/2}}{4x^2} \approx \frac{5}{4}\lim_{x\to\infty}\frac{x^{3/2}}{x^{4/2}} \approx \frac{5}{4}\lim_{x\to\infty}\frac{1}{x^{1/2}} \approx \frac{5}{4}\cdot 0 = 0$$

Ex.4 Find the following three limits:

(a) $\displaystyle\lim_{x\to\infty}\frac{5x^{3/2}}{4x^2+1}$

$$= \lim_{x\to\infty}\left[\frac{5x^{3/2}}{4x^2+1}\right]\left[\frac{\frac{1}{x^{4/2}}}{\frac{1}{x^2}}\right]$$

$$= \lim_{x\to\infty}\frac{\frac{5}{x^{1/2}}}{4+\frac{1}{x^2}}$$

$$= \frac{\lim_{x\to\infty}\frac{5}{x^{1/2}}}{\lim_{x\to\infty}\left(4+\frac{1}{x^2}\right)}$$

$$\to = \frac{0}{4+0} = 0$$

skip?

what not to do:

$$\frac{\frac{5}{\sqrt{\infty}}}{4+\frac{1}{\infty^2}} = \frac{0}{4+0} = 0$$

(b) $\displaystyle\lim_{x\to\infty}\frac{5x^{3/2}}{4x^{3/2}+1}$

$$= \lim_{x\to\infty}\left[\frac{5x^{3/2}}{4x^{3/2}+1}\right]\left[\frac{\frac{1}{x^{3/2}}}{\frac{1}{x^{3/2}}}\right]$$

$$= \lim_{x\to\infty}\frac{5}{4+\frac{1}{x^{3/2}}}$$

$$= \frac{5}{4+0}$$

$$= \frac{5}{4}$$

(c) $\displaystyle\lim_{x\to\infty}\frac{5x^{3/2}}{4\sqrt{x}+1}$

$$= \lim_{x\to\infty}\left[\frac{5x^{3/2}}{4x^{1/2}+1}\right]\left[\frac{\frac{1}{x^{1/2}}}{\frac{1}{x^{1/2}}}\right]$$

Do Not Skip

$$= \lim_{x\to\infty}\frac{5x}{4+\frac{1}{x^{1/2}}}$$

$$= \frac{\lim_{x\to\infty}(5x)}{\lim_{x\to\infty}\left(4+\frac{1}{x^{1/2}}\right)}$$

Increases without Bound

D.N.E.

$$= \frac{\infty}{4+0} = \frac{\infty}{4} = \infty$$

On a Test

Ex.5 Find the following two limits:

a) $$\lim_{x\to\infty} \frac{3x-2}{\sqrt{2x^2+1}}$$

$x \to \infty$
$\sqrt{x^2} = x$

$$= \lim_{x\to\infty} \left[\frac{3x-2}{\sqrt{2x^2+1}}\right]\boxed{\frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}}$$

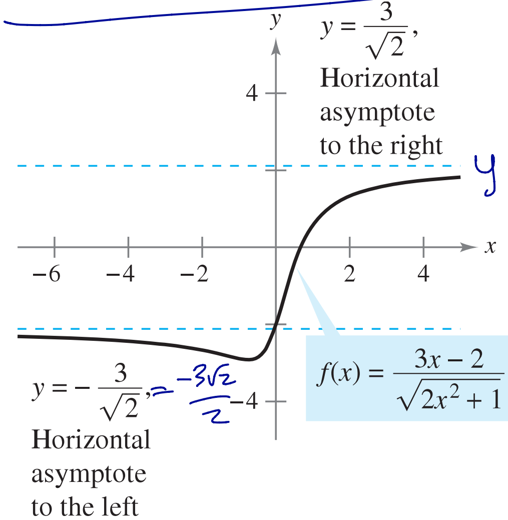
$$= \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}}$$

$$= \frac{\lim_{x\to\infty}\left(3-\frac{2}{x}\right)}{\sqrt{\lim_{x\to\infty}\left(2+\frac{1}{x^2}\right)}}$$

$$= \frac{3-0}{\sqrt{2+0}}$$

$$= \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2}$$



$y = \frac{3\sqrt{2}}{2}$

Functions that are not rational may have different right and left horizontal asymptotes.
Figure 3.38

(b) $$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$$

as $x \to -\infty$
$\sqrt{x^2} = -x$
$\sqrt{(-5)^2} = \sqrt{25} = 5$
$\sqrt{(-5)^2} = -(-5) = 5$

On a Test

$$\lim_{x\to-\infty} \frac{3x-2}{\sqrt{2x^2+1}}$$

$$= \lim_{x\to-\infty} \left[\frac{\frac{3x-2}{1}}{\frac{\sqrt{2x^2+1}}{1}}\right]\boxed{\frac{\frac{1}{-x}}{\frac{1}{\sqrt{x^2}}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{3x}{-x}-\frac{2}{-x}}{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}$$

$$= \frac{\lim_{x\to-\infty}\left(-3+\frac{2}{x}\right)}{\lim_{x\to-\infty}\sqrt{2+\frac{1}{x^2}}}$$

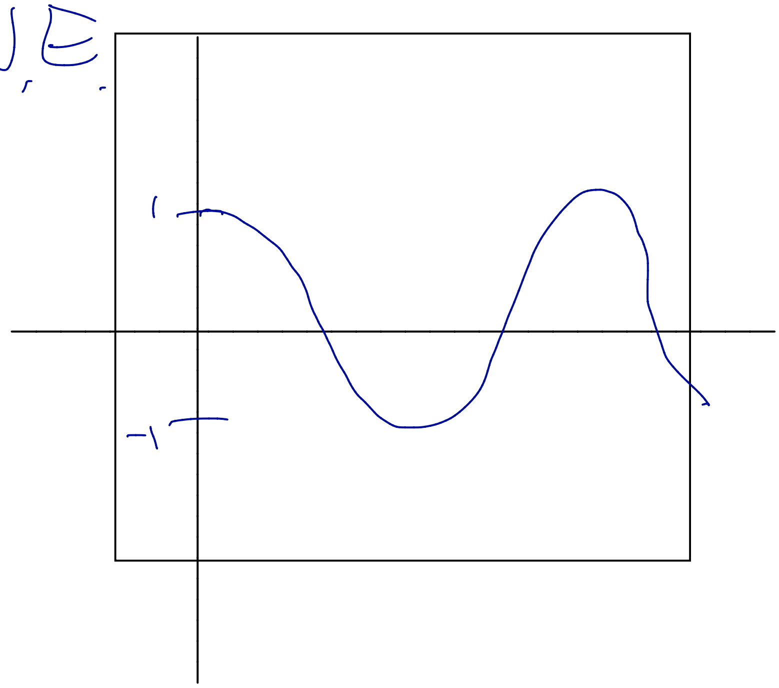
$$= \frac{-3+0}{\sqrt{2+0}}$$

$$y = \frac{-3}{\sqrt{2}}$$

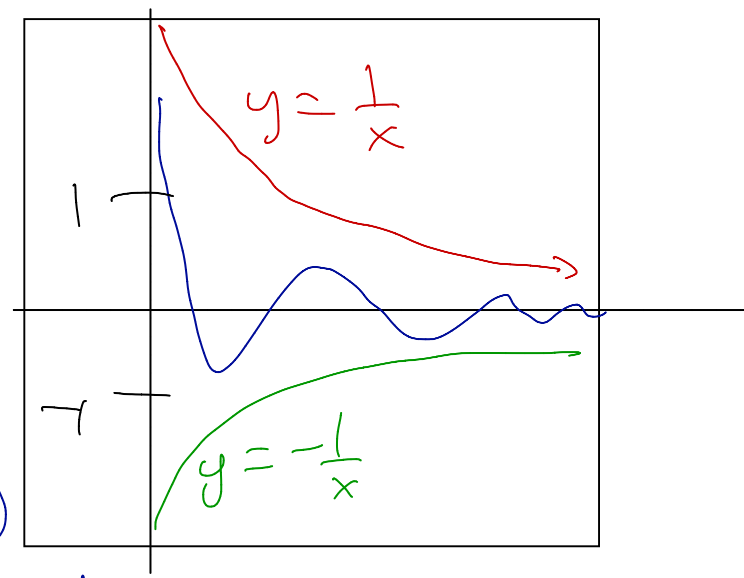
$$y = \frac{-3\sqrt{2}}{2}$$

Ex.6 Find the limit: $\lim\limits_{x\to\infty} \cos(x) = \text{D.N.E.}$

oscillation



Ex.7 Find the limit: $\lim\limits_{x\to\infty} \dfrac{\cos(x)}{x}$



We know

$$-1 \le \cos(x) \le 1 \quad \text{for all } x \ge 0$$

We can

$$\frac{-1}{x} \le \frac{\cos(x)}{x} \le \frac{1}{x} \quad \text{for all } x > 0$$

$x \to \infty$

Consider

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) \le \lim_{x\to\infty} \frac{\cos(x)}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\cos(x)}{x} \le 0$$

So, we have $\lim\limits_{x\to\infty} \dfrac{\cos(x)}{x} = 0$

Infinite Limits at Infinity

Many functions do not approach a finite limit as x increases (or decreases) without bound. For instance, no polynomial function has a finite limit at infinity. The following definition is used to describe the behavior of polynomial and other functions at infinity.

Definition of Infinite Limits at Infinity

Let f be a function defined on the interval (a, ∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each positive number M, there is a corresponding number $N > 0$ such that $f(x) > M$ whenever $x > N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each negative number M, there is a corresponding number $N > 0$ such that $f(x) < M$ whenever $x > N$.

Ex.8 Find the limit: $\lim_{x\to\infty}\left(\frac{1}{2}x + \frac{4}{x^2}\right)$ Explain

$= \lim_{x\to\infty}\left(\frac{1}{2}x\right) + \lim_{x\to\infty}\frac{4}{x^2}$

$= \infty + 0$

$= \infty$

DNE.

Increases without Bound